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Finite element analysis of creep in unidirectional composites based on homogenization theory

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Abstract—The creep behavior of continuous fiber reinforced unidirectional composites due to the viscoelasticity of the resin matrix is analyzed based on the homogenization theory utilizing the finite element method. It is assumed that the constituent fiber is transversely isotropic linearly elastic material and that the matrix is isotropic linearly elastic and nonlinearly viscoelastic. The theoretical predictions for the creep behavior of the composites are compared with the experimental results.

Keywords: Fiber reinforced composite; creep; homogenization theory; finite element method.

1. INTRODUCTION

Fiber reinforced plastics exhibit creep behavior due to the viscoelasticity of the resin matrix even at room temperature. Since a variety of composites are being considered for use in aerospace structures at high temperature, it is critical to predict the creep behavior of composites. Experimental testing of the creep behavior of composites is very difficult and time-consuming to conduct. In contrast to isotropic materials, for which measurement of creeping can be performed independent of direction, composites which are anisotropic require characterizing creep behavior in different directions relative to the fibers. Thus, development of reliable models for predicting the time-dependent characteristic of composites has received significant attention in the literature. These studies involve numerical models [1–3] and analytical models [4–6]. The creep of unidirectionally reinforced composites were numerically analyzed by the finite element method utilizing the repeating unit in composites with regular fiber packing [1] or the homogenization theory [2, 3].

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The phenomenological model [4] was evaluated by the experimental data while the theoretical models [5, 6] were verified by the finite element calculations.

In the present paper, the creep of a unidirectional composite with regular fiber arrangement is analyzed by the finite element method based on the homogenization theory. It is assumed that the constituent fiber is transversely isotropic linearly elastic material while the matrix is isotropic linearly elastic and nonlinearly viscoelastic.

2. GOVERNING EQUATIONS

We deal with the incremental deformation of a periodic nonhomogeneous material. The equilibrium equations are given by

$$\frac{\partial \Delta\sigma_{ij}}{\partial x_j} = 0, \tag{1}$$

and the strain displacement relationship is written as

$$\Delta\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \Delta u_i}{\partial x_j} + \frac{\partial \Delta u_j}{\partial x_i}\right). \tag{2}$$

The constitutive equations take the form

$$\Delta\sigma_{ij} = C_{ijk\ell}\left[\Delta\varepsilon_{k\ell} - \dot{\varepsilon}^c_{k\ell}(\sigma_{t+\theta\Delta t})\Delta t\right], \tag{3}$$

where $\dot{\varepsilon}^c_{k\ell}(\sigma_{t+\theta\Delta t})$ is the creep strain rate at $t = t + \theta\Delta t$ $(0 \leqslant \theta \leqslant 1)$ and $C_{ijk\ell}$ represents the elastic constant.

The stress boundary conditions are given by

$$\Delta\sigma_{ij}n_j = \Delta\bar{T}_i \quad \text{on } \Omega_{\mathrm{T}}, \tag{4}$$

while the displacement boundary conditions are expressed as

$$\Delta u_i = \Delta\bar{u}_i \quad \text{on } \Omega_{\mathrm{U}}. \tag{5}$$

3. HOMOGENIZATION METHOD

3.1. Global and local coordinates

We define the local coordinates $\mathbf{y}$ in a single repetitive cell of periodicity as

$$\mathbf{y} = \frac{\mathbf{x}}{\varepsilon} \qquad (\varepsilon \ll 1), \tag{6}$$

where the global coordinates $\mathbf{x}$ refer to the whole of the body. By using the rule of differentiation

$$\frac{d}{dx_j} = \frac{\partial}{\partial x_j} + \frac{\partial y_k}{\partial x_j}\frac{\partial}{\partial y_k} = \frac{\partial}{\partial x_j} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_j}, \tag{7}$$

the equilibrium equations as equations (1) are reduced to

$$\frac{\partial \Delta\sigma_{ij}}{\partial x_j} + \frac{1}{\varepsilon}\frac{\partial \Delta\sigma_{ij}}{\partial y_j} = 0, \tag{8}$$

and the strain–displacement relationship as equations (2) are transformed to

$$\Delta\varepsilon_{ij} = \varepsilon_{x(ij)}(\Delta\mathbf{u}) + \frac{1}{\varepsilon}\varepsilon_{y(ij)}(\Delta\mathbf{u}) \tag{9}$$

where

$$\varepsilon_{x(ij)}(\Delta\mathbf{u}) = \frac{1}{2}\left(\frac{\partial \Delta u_j}{\partial x_i} + \frac{\partial \Delta u_i}{\partial x_j}\right), \quad \varepsilon_{y(ij)}(\Delta\mathbf{u}) = \frac{1}{2}\left(\frac{\partial \Delta u_j}{\partial y_i} + \frac{\partial \Delta u_i}{\partial y_j}\right) \tag{10}$$

3.2. Asymptotic expansion

We assume that the periodicity of material characteristics imposes an analogous periodical perturbation on quantities describing the mechanical behavior of the body. Hence we use asymptotic expansion to describe the displacements and stresses as

$$u_i(\mathbf{x}, \mathbf{y}) = u_i^0(\mathbf{x}) + \varepsilon u_i^1(\mathbf{x}, \mathbf{y}) + \varepsilon^2 u_i^2(\mathbf{x}, \mathbf{y}) + \cdots, \tag{11}$$

$$\sigma_{ij}(\mathbf{x}, \mathbf{y}) = \sigma_{ij}^1(\mathbf{x}, \mathbf{y}) + \varepsilon\sigma_{ij}^2(\mathbf{x}, \mathbf{y}) + \cdots, \tag{12}$$

where u_i^k, σ_{ij}^{k+1} for $k \geqslant 1$ take the same value on the opposite sides of the cell of periodicity (i.e. Y-periodic). Substituting equation (12) into equation (8), the equilibrium equation splits into terms of orders ε^{-1} and ε^0 as

$$\frac{\partial \Delta\sigma_{ij}^1}{\partial y_j} = 0, \tag{13}$$

$$\frac{\partial \Delta\sigma_{ij}^1}{\partial x_j} + \frac{\partial \Delta\sigma_{ij}^2}{\partial y_j} = 0. \tag{14}$$

Introduction of equation (11) into equation (9) yields

$$\Delta\varepsilon_{ij} = \varepsilon_{x(ij)}\big(\Delta\mathbf{u}^0(\mathbf{x})\big) + \varepsilon_{y(ij)}\big(\Delta\mathbf{u}^1(\mathbf{x},\ \mathbf{y})\big) + \varepsilon\big[\varepsilon_{x(ij)}\big(\Delta\mathbf{u}^1\big) + \varepsilon_{y(ij)}\big(\Delta\mathbf{u}^2\big)\big] + \cdots. \tag{15}$$

The creep strain rate $\dot{\varepsilon}_{k\ell}(\sigma_{t+\theta\Delta t})$ in equation (3) can be approximated by

$$\dot{\varepsilon}_{k\ell}^c(\sigma_{t+\theta\Delta t}) = \dot{\varepsilon}_{k\ell}^c(\sigma_t) + \theta\frac{\partial \dot{\varepsilon}_{k\ell}^c(\sigma_t)}{\partial \sigma_{rs}}\Delta\sigma_{rs} \quad (0 \leqslant \theta \leqslant 1) \tag{16}$$

Substituting equation (12) into equation (16), we get

$$\dot{\varepsilon}_{k\ell}^c(\sigma_{t+\Theta\Delta t}) = \dot{\varepsilon}_{k\ell}^c\big(\sigma_t^1\big) + \dot{G}_{k\ell rs}\big(\sigma_t^1\big)\big\{\theta\Delta\sigma_{rs}^1 + \varepsilon\left(\sigma_{rs}^2 + \theta\Delta\sigma_{rs}^2\right) + \cdots\big\} \tag{17}$$

where

$$\dot{G}_{k\ell rs}(\sigma) = \frac{\partial \dot{\varepsilon}_{ke}^c(\sigma)}{\partial \sigma_{rs}}. \tag{18}$$

By introducing equations (12), (15), (17) into equation (3), the constitutive equation splits into terms of order ε^0 and ε^1 as

$$\begin{aligned}\Delta\sigma_{ij}^1 &= \left[\left(C_{ijk\ell}\right)^{-1} + \theta\dot{G}_{k\ell ij}\left(\sigma_t^1\right)\Delta t\right]^{-1}\left[\varepsilon_{x(k\ell)}\left(\Delta\mathbf{u}^0\right) + \varepsilon_{y(k\ell)}\left(\Delta\mathbf{u}^1\right) - \dot{\varepsilon}_{k\ell}^c\left(\sigma_t^1\right)\Delta t\right] \\ &= \tilde{C}_{ijk\ell}\left[\varepsilon_{x(k\ell)}\left(\Delta\mathbf{u}^0\right) + \varepsilon_{y(k\ell)}\left(\Delta\mathbf{u}^1\right) - \dot{\varepsilon}_{k\ell}^c\left(\sigma_t^1\right)\Delta t\right], \end{aligned} \tag{19}$$

$$\Delta\sigma_{ij}^2 = \tilde{C}_{ijk\ell}\left[\varepsilon_{x(k\ell)}\left(\Delta\mathbf{u}^1\right) + \varepsilon_{y(k\ell)}\left(\Delta\mathbf{u}^2\right) - \dot{G}_{k\ell rs}\left(\sigma_t^1\right)\sigma_{rs}^2\Delta t\right]. \tag{20}$$

3.3. Microscopic and macroscopic equations

We decompose the displacement increment in the perturbation into the elastic and creep parts as

$$\Delta u_i^1 = \Delta u_i^{1e} + \Delta u_i^{1c} = \chi_i^{k\ell}(\mathbf{y})\varepsilon_{x(k\ell)}(\Delta\mathbf{u}_0(\mathbf{x})) + \Delta u_i^{1c}, \tag{21}$$

where $\chi_i^{k\ell}$ and Δu_i^{1c} are Y-periodic. Substituting equation (21) into equation (19), we get the macroscopic evolution equation as

$$\begin{aligned}\Delta\sigma_{ij}^1 &= \tilde{C}_{ijpq}\left[\delta_{pk}\delta_{q\ell} + \varepsilon_{y(pq)}\left(\chi^{k\ell}\right)\right]\varepsilon_{x(k\ell)}\left(\Delta\mathbf{u}^0\right) + \tilde{C}_{ijk\ell}\left[\varepsilon_{y(k\ell)}\left(\Delta\mathbf{u}^{1c}\right) - \dot{\varepsilon}_{k\ell}^c\left(\sigma_t^1\right)\Delta t\right] \\ &= \tilde{a}_{ijk\ell}\varepsilon_{x(k\ell)}\left(\Delta\mathbf{u}^0\right) + \Delta r_{ij}. \end{aligned} \tag{22}$$

Here, we introduce a volume average operator

$$\langle\cdot\rangle = \frac{1}{|\mathbf{y}|}\int_{\mathbf{y}}(\cdot)\ \mathrm{d}\mathbf{y}, \tag{23}$$

where $|\mathbf{y}|$ denotes volume of the unit cell $\mathbf{y}$. From equation (22), we get the macroscopic constitutive equations

$$\Delta\Sigma_{ij} = \left\langle\tilde{a}_{ijk\ell}\right\rangle\Delta E_{k\ell} + \left\langle\Delta r_{ij}\right\rangle, \tag{24}$$

where

$$\Delta\Sigma_{ij} = \left\langle\Delta\sigma_{ij}\right\rangle, \quad \Delta E_{k\ell} = \varepsilon_{x(k\ell)}\left(\Delta\mathbf{u}^0\right). \tag{25}$$

Introducing equation (22) into equation (13) and considering that $\varepsilon_{x(k\ell)}(\Delta\mathbf{u}^0)$ are arbitrary, we get

$$\frac{\partial\tilde{a}_{ijk\ell}}{\partial y_j} = 0, \tag{26}$$

$$\frac{\partial\Delta r_{ij}}{\partial y_j} = 0, \tag{27}$$

which give the governing equations for χ^{pq} and $\Delta\mathbf{u}^{1c}$, respectively, as

$$\frac{\partial}{\partial y_j}\left\{\tilde{C}_{ijk\ell}\left[\delta_{kp}\delta_{\ell q} + \varepsilon_{y(k\ell)}\left(\chi^{pq}\right)\right]\right\} = 0, \tag{28}$$

$$\frac{\partial}{\partial y_j}\left\{\tilde{C}_{ijk\ell}\left[\varepsilon_{y(k\ell)}(\Delta\mathbf{u}^{1c}) - \dot{\varepsilon}_{k\ell}^c\left(\sigma_t^1\right)\Delta t\right]\right\} = 0. \tag{29}$$

Taking the volume average of equation (14), we obtain the macroscopic equilibrium equations as

$$\frac{\partial \Delta \Sigma_{ij}}{\partial x_j} = 0. \tag{30}$$

4. CREEP LAW

The flow rule of creep deformation of the isotropic matrix is

$$\dot{\varepsilon}_{ij}^{c} = \frac{3}{2}\frac{\dot{\varepsilon}_{e}^{c}}{\sigma_e}\sigma_{ij}', \tag{31}$$

where σ_e and $\dot{\varepsilon}_e^c$ are the effective stress and the effective creep strain increment, respectively, in the form

$$\sigma_e = \left(\frac{3}{2}\sigma_{ij}'\sigma_{ij}'\right)^{1/2}, \quad \dot{\varepsilon}_e^c = \left(\frac{2}{3}\dot{\varepsilon}_{ij}^c\dot{\varepsilon}_{ij}^c\right)^{1/2}. \tag{32}$$

It is assumed that the creep of matrix is governed by the Norton–Bailey law as

$$\varepsilon_e^c = k\sigma_e^n t^m, \tag{33}$$

from which we have the creep law of the time-hardening type in the form

$$\dot{\varepsilon}_e^c = km\sigma_e^n t^{m-1}. \tag{34}$$

Eliminating t from equations (33) and (34), we obtain the creep law of the strain-hardening type as

$$\dot{\varepsilon}_e^c = k^{\frac{1}{m}} m \sigma_e^{\frac{n}{m}} \varepsilon_e^{c\,\frac{m-1}{m}}. \tag{35}$$

5. FINITE ELEMENT FORMULATION

We solve equations (28) and (29) utilizing the finite element method. Considering that χ_i^{pq} is Y-periodic, the Galerkin equation for equation (28) can be reduced to

$$\begin{aligned} 0 &= \int_Y \frac{\partial}{\partial y_j}\left\{\tilde{C}_{ijk\ell}\left[\delta_{kp}\delta_{\ell q} + \varepsilon_{y(k\ell)}\left(\chi^{pq}\right)\right]\right\}\delta\chi_i^{pq}\,\mathrm{d}Y \\ &= -\int_Y \tilde{C}_{ijpk}\frac{\partial\delta\chi_i^{pq}}{\partial y_j}\,\mathrm{d}Y - \int_Y \tilde{C}_{ijk\ell}\varepsilon_{y(k\ell)}\left(\chi^{pq}\right)\frac{\partial\delta\chi_i^{pq}}{\partial y_j}\,\mathrm{d}Y, \end{aligned} \tag{36}$$

which is the weak form utilized to obtain χ_i^{pq} in the finite element method. The Galerkin equation for equation (29) can also be transformed to

$$\begin{aligned} 0 &= \int_Y \frac{\partial}{\partial y_j}\left\{\tilde{C}_{ijk\ell}\left[\varepsilon_{y(k\ell)}\left(\Delta\mathbf{u}^{1c}\right) - \dot{\varepsilon}_{k\ell}^c\left(\sigma_t^1\right)\Delta t\right]\right\}\delta\Delta u_i^{1c}\,\mathrm{d}Y \\ &= -\int_Y \tilde{C}_{ijpk}\varepsilon_{y(k\ell)}\left(\Delta\mathbf{u}^{1c}\right)\frac{\partial\delta\Delta u_i^{1c}}{\partial y_j}\,\mathrm{d}Y - \int_Y \tilde{C}_{ijk\ell}\dot{\varepsilon}_{k\ell}^c\left(\sigma_t^1\right)\Delta t\frac{\partial\delta\Delta u_i^{1c}}{\partial y_j}\,\mathrm{d}Y, \end{aligned} \tag{37}$$

which is the weak form to obtain Δu_i^{1c} for the periodic boundary conditions.

6. NUMERICAL PROCEDURE

For the case where macroscopic stresses Σ_{ij} (or macroscopic strains E_{ij}) are given, we set $\sigma_t^1 = 0$ for $t = 0$. Then the incremental computation from the current time $t = t$ to the subsequent time $t = t + \Delta t$ is conducted as follows:

(1) For given σ_t^1, we calculate $\tilde{C}_{ijk\ell}$ from equation (19), and $\dot{\varepsilon}^c_{k\ell}(\sigma_t^1)$ from equation (31). And we obtain χ_i^{pq} and Δu_i^{1c} by the finite element method based on equations (36) and (37) with the Y-periodic boundary conditions.

(2) We calculate $\tilde{a}_{ijk\ell}$ and Δr_{ij} from equation (22), and obtain $\langle \tilde{a}_{ijk\ell} \rangle$ and $\langle \Delta r_{ij} \rangle$.

(3) We calculate $\Delta E_{ij} = \dot{\varepsilon}^c_{k\ell}(\Delta \mathbf{u}_0)$ (or $\Delta \Sigma_{ij}$) from equation (24).

(4) We calculate $\Delta \sigma_{ij}^1$ from equation (22), and obtain σ_t^1 at $t = t + \Delta t$.

7. NUMERICAL RESULTS

We analyze a unidirectional graphite-epoxy composite with hexagonal packing as shown in Fig. 1 with θ in equation (3) being $1/2$. The epoxy resin is considered to be isotropic linearly elastic and nonlinearly viscoelastic material with the mechanical properties

$$E_m = 476\ \text{kgf/mm}^2, \quad \nu_m = 0.39, \quad n = 1, \quad m = 0.269,$$
$$k = 1.76 \times 10^{-4}\ (\text{kgf/mm}^2)^{-n}(\text{hour})^{-m},$$

which were determined by the experiment conducted on the epoxy resin [1]. The graphite fiber is assumed to be transversely isotropic linearly elastic material with

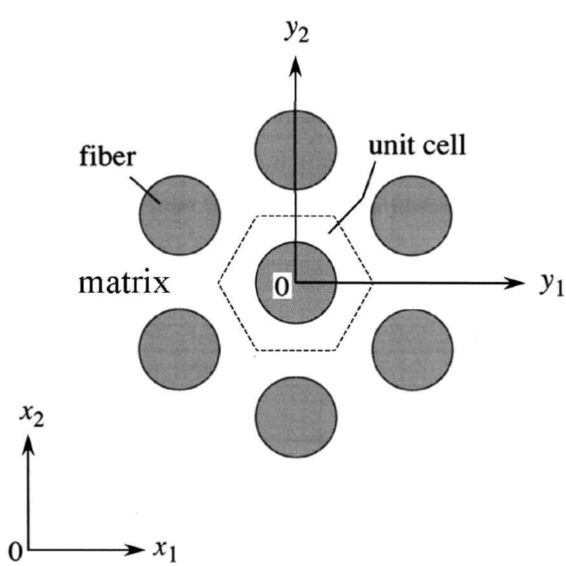

Figure 1. Regular hexagonal array composite and unit cell.

the mechanical properties

$$E_{Lf} = 21000\,\text{kgf/mm}^2, \quad E_{Tf} = 1310\,\text{kgf/mm}^2,$$
$$\nu_{LTf} = 0.29, \quad \nu_{LTf} = 0.49, \quad G_{LTf} = 2190\,\text{kgf/mm}^2.$$

A unit cell of periodicity as indicated in Fig. 1 is discretized by the eight-node isoparametric quadrilateral finite elements as shown in Fig. 2.

The macroscopic strain E_{11} and the macroscopic creep strain E_{11}^c of the graphite-epoxy unidirectional composite subjected to unidirectional constant macroscopic stress $\Sigma_{11} = \bar{\sigma}$ are shown as a function of t in Figs 3 and 4, respectively, together with the experimental results [1]; and the macroscopic strain E_{22} *vs.* the time t is shown in Fig. 5.

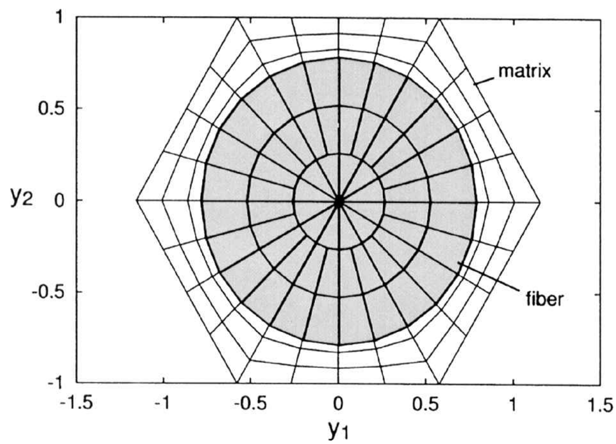

Figure 2. Finite element discretization of unit cell of hexagonal array composite.

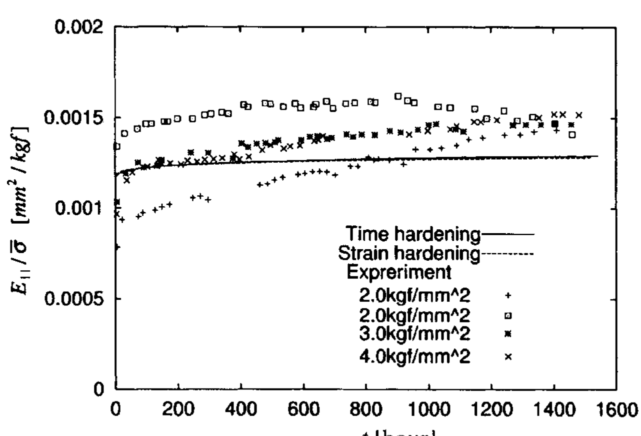

Figure 3. Strain E_{11} vs. time t for unidirectional graphite-epoxy composite under applied stress $\Sigma_{11} = \bar{\sigma}$.

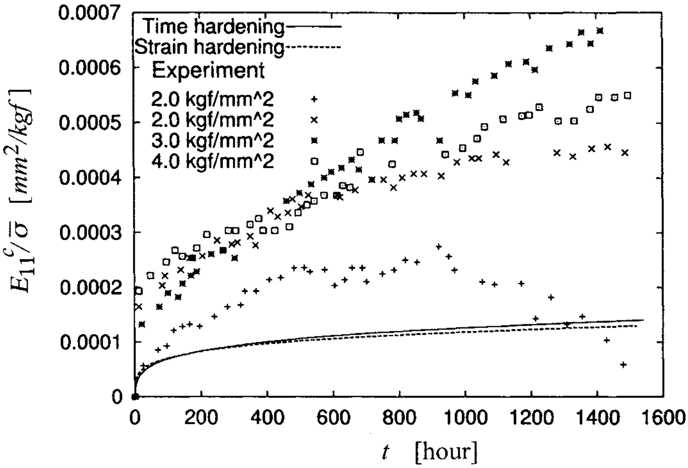

Figure 4. Creep strain E_{11}^{c} vs. time t for unidirectional graphite-epoxy composite under applied stress $\Sigma_{11} = \bar{\sigma}$.

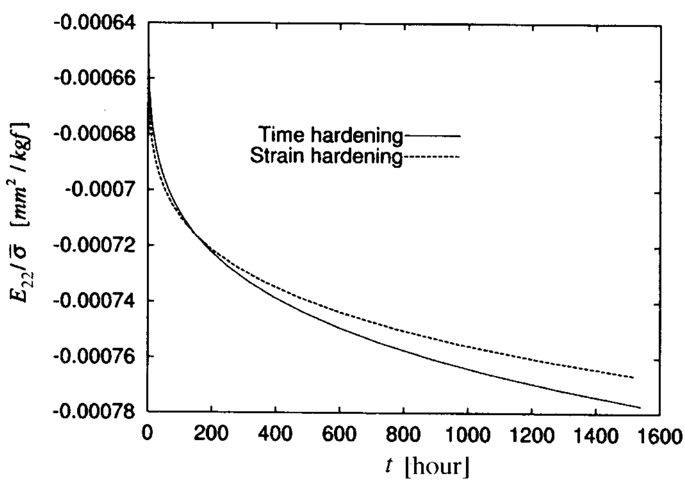

Figure 5. Strain E_{22} vs. time t for unidirectional graphite-epoxy composite under applied stress $\Sigma_{11} = \bar{\sigma}$.

Since we assume that the epoxy resin is linearly viscoelastic ($n = 1$), the total compliance $E_{11}/\bar{\sigma}$ and the creep compliance $E_{11}^{c}/\bar{\sigma}$ and $E_{22}^{c}/\bar{\sigma}$ are the same for any applied stress $\bar{\sigma}$. It can be seen from Fig. 4 that the creep compliance $E_{11}^{c}/\bar{\sigma}$ for the strain-hardening law is larger than for the time-hardening law at the first transient stage and becomes smaller at the later stationary stage. The Poisson shrinkage ($-E^{22}$) in Fig. 5 shows the same characteristic for the strain-hardening law and the time-hardening law. In view of Fig. 3, the experimental results for the total compliance qualitatively agree with the theoretical prediction. However, the experimental data for the creep compliance are much higher than the theoretical results.

8. CONCLUSIONS

Creep behavior of unidirectional fiber reinforced plastics is analyzed by the finite element method utilizing the homogenization theory. To verify the theoretical predictions, more extensive creep tests on composites and the constituent resins should be conducted. Although composites under uniform macroscopic strains have been analyzed, the present homogenization method can easily be extended to study composites subjected various boundary conditions.

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